MERU UNIVERSITY OF SCIENCE AND TECHNOLOGY

P.O. Box 972-60200 – Meru-Kenya.

Tel: +254 (0)799529958, +254 (0)799529959, +254 (0)712524293

Website: www.must.ac.ke Email: info@must.ac.ke

University Examinations 2022/2023

FIRST YEAR, FIRST SEMESTER EXAMINATION FOR THE DOCTOR OF PHILOSOPHY IN MATHEMATICS

SMA 8116: FUNCTIONAL ANALYSIS

DATE: APRIL 2023 **TIME: 3 HOURS**

INSTRUCTIONS: *Answer any three questions*

QUESTION ONE (20 MARKS)

a) (i) Define a norm in a vector space X (3 marks)

(ii) What is a normed linear space? (2 marks)

b) (i) Define a Cauchy sequence in a metric space (X, d) (2 marks)

(ii) Prove that every convergent sequence in a metric space is Cauchy (3 marks)

c) (i) Give two examples of a metric space (2 marks)

(ii) Show that $d(x, y) = (x - y)^2$ define a metric on the set of all real numbers $\mathbb{R}$ (3 marks)

d) Let X be the set of all ordered triplex of zeros and ones. Like all the elements of X and state a possible metric on X. (5 marks)

QUESTION TWO (20 MARKS)

a) Explain fully what a linear operator T is (3 marks)

Give three examples of linear operators (3 marks)

b) Let X and Y be two linear spaces over a scalar field K, and let $T : X \to Y$ be a linear map

(i) Prove that T is one –to-one if and only if $T(x) = 0 \implies x = 0$ (4 marks)

(ii) If T one-to-one, then T^{-1} exists on $R\ (T)$ and $T^{-1}: R(T) \rightarrow X$ is a linear map (4 marks)

(iii) Explain the continuity of the inner product (3 marks)

QUESTION THREE (20 MARKS)

a) (i) Define a Banach space V (2 marks)

(ii) Define Hilbert space H (2 marks)

(iii) For all $I \leq P \leq \infty$ we have ι^P with respect to the norm $\|\cdot\|$ is a Banach space. Prove (9 marks)

b) Let V be an inner product space for each $V \in V$, let $\|V\| = \sqrt{< V, V >}$ and $x, y \in V$, prove that $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$,the parallelogram rule (7 marks)

QUESTION FOUR (20 MARKS)

a) Distinguish between $L^p(u)$ and ι^P as vector spaces (4 marks)

b) (i) Suppose (X, S, μ) is a measure space, $1 \leq P \leq \infty$ and $f, h : X \rightarrow F$ one $S -$ measureable.

(ii) State and prove the Holder's inequality (8 marks)

c) Explain what is meant by an open unit disc D in the real plane (3 marks)

Given the Bergman space $L^P_\propto(D)$, for $z \in D$ explain the Bergman reproducing Kernel K_Z (5 marks)

QUESTION FIVE (20 MARKS)

a) Explain the following terms

(i) Analytic function (2 marks)

(ii) Bloch space (2 marks)

(iii) Besov space (2 marks)

(iv) Mobius – invariant of a measure (2 marks)

b) Let $I \leq P \leq \infty$, $\propto > 0$ and $0 < v < 1$. There is a constant C, only depending upon $P, \propto$ $and\ r$ such that

$\int_{D(w,r)} |g(z)|^P \left(|-1z|^2\right)^\alpha \ dT(z) \geq C|g(w)|^P \left(|-1w|^2\right)^\alpha$ for every analytic function g on D and $w \in D$. Prove (12 marks)