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University Examinations 2022/2023

FIRST YEAR, FIRST SEMESTER EXAMINATION FORTHE DOCTOR OF PHILOSOPHY IN MATHEMATICS

SMA 8114: DIFFERENTIAL GEOMETRY

DATE: APRIL 2023 **TIME: 3 HOURS**

INSTRUCTIONS: *Answer question **one** and any other **two** questions*

QUESTION ONE (30 MARKS)

a) Let $\alpha : I \to \mathbb{R}^3$ be a curve and let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Prove that $L_a^b(\alpha) = L_a^b(\varphi \circ \alpha)$. That is, rigid motions preserve the length of curves (5 marks)

b) Let $I \to \mathbb{R}^3$ be a regular curve (not necessarily p.b.a.1), show that the curvature and torsion of α at t ϵ I is given respectively by (5 marks)

$$k(t) = \frac{|\alpha'(t)\, x\, a''(t)|}{|a'(t)|^3} \quad \text{and } \tau(t) = -\frac{\left(\alpha'(t)\, x\, \alpha''(t)\right).\alpha'''(t)}{|\alpha'(t)\ x\, \alpha''(t)|^2}$$

c) (*Invariance under rigid motions*) Let $S \subset \mathbb{R}^3$ be an orientable surface and $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of $\mathbb{R}^3$, i.e $\varphi(p) = Ap + b$ for some $A \ \epsilon\ O(3)$ and $b \,\epsilon\, \mathbb{R}^3$. Let $S'(0) = \varphi(S)$ be the image surface of S under φ.

i. If $N : S \to \mathbb{S}^2$ Is a Gauss map for S, prove that $N' = A(N \circ \varphi^{-1}) : S' \to \mathbb{S}^2$ is a Gauss map for S' (5 marks)

ii. Let A and A' be the second fundamental for S and S' respectively (with respect to N and N' in (i). Show that for any $p \ \epsilon\ S$ and $v, w \ \epsilon\ T_P S$, $A'_{\varphi(p)}\left(d\varphi p(v), d\varphi p(w)\right) = A_p(v, w)$. (5 marks)

iii. Find the relation between the mean curvature and Gauss curvature of S and S'.

(2 marks)

d) Find a local isometry $f : S_1 \to S_2$ from the upper half place $S_1 = \{z = 0, y > 0\}$ to the cone $S_2 := \{x^2 + y^2 = z^2, z > 0\}$. Calculate the mean and Gauss curvatures of S_2. (5 marks)

e) Prove that the surfaces parametrized by $(u, v) \epsilon (0, +\infty) x (0, 2\pi)$,

$$X(u, v) = (u \cos v, u \sin v, \log u)$$

$$X^{\sim}(u, v) = (u \cos v, u \sin v, v)$$

Have the same Gauss curvature at the points $X(u, v)$ and $\widetilde{X}(u, v)$. However, show that the map $X^{\sim} \circ X^{-1}$ is not an isometry (3 marks)

QUESTION TWO (15 MARKS)

a) Let $\alpha(s) : [0, L] \to \mathbb{R}^2$ be a simple closed convex plane curve p.b.a. I which is oriented in the counterclockwise direction. Let $\{T(s), N(s)\}$ be the Frenet frame of $\alpha(s)$. The curve $\beta(s): [0, L] \to \mathbb{R}^2$ given by $\beta(s) := \alpha(s) - rN(s)$, whether $r > 0$ is a constant, is called a *parallel curve* to α. Show that

(i) $Lenght\ (\beta) = Lenght\ (\alpha) + 2\pi r$ (4 marks)

(ii) $Area\ (\Omega_\beta) = Area\ (\Omega_\alpha) + rL + \pi r^2$ (5 marks)

where Ω_α, Ω_β are the regions bounded by α and β respectively.

b) If $S \subset \mathbb{R}^3$ is a closed surface with positive Gauss curvature $K > 0$, show that any two simple closed geodesics on S must intersect. (6 marks)

QUESTION THREE (20 MARKS)

a) Let p be a point on a surface $S \subset \mathbb{R}^3$.

(i) Prove that $K(p) > 0$ if and only if there exists a point $p_o \epsilon \mathbb{R}^3$ such that p is a local maximum of the function $f(x) = |x - p_o|^2$. (6 marks)

(ii) Show that there is not compact surface $S \subset \mathbb{R}^3$ with $K > 0$ everywhere (4 marks)

b) Find a local isometry between the helicoid and the catenoid. Are they globally isometric?

(5 marks)

QUESTION FOUR (20 MARKS)

a) Let $S \subset \mathbb{R}^3$ be a surface

(i) Fix a point $p_o \epsilon \mathbb{R}^3$ such that $p_o \neq S$, show that the *distance function from p* 0 $f(p) := |p - p_0|$ Defines a smooth function $f : S \to R$. Moreover, prove that $p \epsilon S$ is a critical point of f if and only if the line joining p and p_o is normal to S at p

(3 marks)

(ii) Fix a unit vector $v \epsilon \mathbb{R}^3$, show that the *height function along v* $h(p): = \langle p, v\rangle$, defines a smooth function $h: S \to R$. Prove that $p \epsilon S$ is a critical point of h if an only if v is normal to S at p. (3 marks)

b) Show that an *ellipsoid* $S = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$, where $a, b, c, > 0$ are constants, has positive Gauss curvature at every point (9 marks)

QUESTION FIVE (20 MARKS)

a) Consider the *logarithmic spiral* $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (ae^{bt} \cos t,\ ae^{bt} \sin t)$ with $a > 0,\ b < 0$. Compute the arc length functions $S : \mathbb{R} \to \mathbb{R}$ form $t_0 \in \mathbb{R}$. Reparametrize this curve by arc length and study its trace (5 marks)

b) Does there exist a parametrization $X(u, v) : \cup \to \mathbb{R}^3$ of a surface S such that the first and second fundamental forms are given by:

(a) $(gij) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(hij) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; (5 marks)

(b) $(gij) = \begin{pmatrix} 1 & 0 \\ 0 & cos^2u \end{pmatrix}$ and $(hij) = \begin{pmatrix} cos^2u & 0 \\ 0 & 1 \end{pmatrix}$? (5 marks)

Does there exist a parametrization $X(u, v) : \cup \to \mathbb{R}^3$ of a surface S such that the first and second fundamental forms are given by:

(a) $(gij) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $(hij) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

(b) $(gij) = \begin{pmatrix} 1 & 0 \\ 0 & cos^2u \end{pmatrix}$ and $(hij) = \begin{pmatrix} cos^2u & 0 \\ 0 & 1 \end{pmatrix}$?